Part II

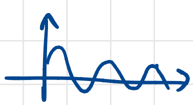
Input-Output stability: (Ch. 5)

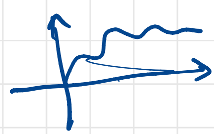
$$\dot{x} = f(t, x, u), \quad x(0) = x_0$$
$$y = h(t, x, u) \qquad (*)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$

- the system $(*)$ defines an operator $\mathcal{H}$ from input signal $(u(t))_{t \geq 0}$ to output $(y(t))_{t \geq 0}$

$$u \longrightarrow \boxed{H} \longrightarrow y = Hu$$





Objective: Study input-output stability:

"well-behaved" input leads to "well-behaved" output

- Space of signals:

$$L_\infty^m = \left\{ u : [0,\infty) \to \mathbb{R}^m \;\middle|\; u \text{ is piecewise cont. and } \|u\|_{L_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty \right\}$$

$$L_2^m = \left\{ u : [0,\infty) \to \mathbb{R}^m \;\middle|\; u \text{ is piecewise cont. and } \|u\|_{L_2} = \left[\int_0^\infty \|u(t)\|^2 dt \right]^{\frac{1}{2}} < \infty \right\}$$

more generally

$$L_p^m = \left\{ u : [0,\infty) \to \mathbb{R}^m \;\middle|\; u \text{ is piecewise cont. and } \|u\|_{L_p} = \left[\int_0^\infty \|u(t)\|^p dt \right] \right\}$$

Remark on notation:

- L_p-Norm of function $u \in L_p^m$:

$$\|u\|_{L_p} = \left[\int_0^\infty \|u(t)\|^p dt \right]^{\frac{1}{p}}$$

p-norm of vector $x \in \mathbb{R}^n$

$$\|x\|_p = \left[x_1^p + \cdots + x_n^p \right]^{\frac{1}{p}}$$

we usually drop p and write $\|x\|$ for simplicity

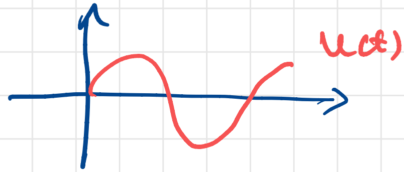
- the norm $\|\cdot\|$ here can be any p-norm of vector

for example $\|u\|_{L_1} = \int_0^\infty \|u(t)\|_2 dt$ or $\|u\|_{L_1} = \int_0^\infty \|u(t)\|_\infty dt$

- They give different values, but equivalent, in the sense that if one is finite, the other one is finite too.

Example:

① $u(t) = \sin(t)$,



$$\|u\|_{L_\infty} = \sup_{t \geq 0} |u(t)| = \sup_{t \geq 0} |\sin(t)| = 1 < \infty$$

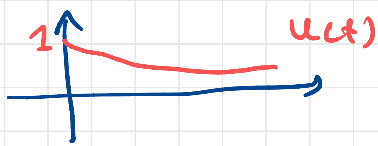
$$\Rightarrow u \in L_\infty^1$$

but

$$\|u\|_{L_2} = \left[\int_0^\infty |\sin(t)|^2 \, dt\right]^{\frac{1}{2}} = \infty$$

$$\Rightarrow u \notin L_2^1$$

② $u(t) = e^{-t}$



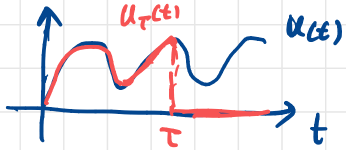
$$\Rightarrow \|u\|_{L_\infty} = 1$$

$$\|u\|_{L_2} = \left[\int_0^\infty e^{-2t} \, dt\right]^{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow u \in L_\infty^1 \text{ and } u \in L_2^1$$

Def: for a function $u:[0,\infty)\to\mathbb{R}^m$, u_τ denotes its truncation

$$u_\tau(t)=\begin{cases} u(t) & 0\le t\le\tau \\ 0 & t>\tau \end{cases}$$



Def: Extended space

$$L_e^m \leftarrow L_{pe}^m = \{u:[0,\infty)\to\mathbb{R}^m \mid u_\tau\in L_p^m \ \forall\tau\ge 0\}$$

L_e^m for short.

- L_{pe}^m is larger than L_p^m, $L_p^m\subseteq L_{pe}^m$

for example, $u(t)=t$ then $u\in L_{pe}^1$

but $u\notin L_p^1$

L-Stability:

Def: The operator $H: L_e^m \to L_e^q$ is L-stable if

$$\|(Hu)_\tau\|_L \leq \alpha(\|u_\tau\|_L) + \beta \qquad \forall u \in L_e^m, \ \forall \tau \geq 0$$

where β is a positive constant and $\alpha: [0,\infty) \to [0,\infty)$ is a strictly increasing function s.t. $\alpha(0) = 0$ $\rightsquigarrow$ class $\mathcal{K}$

- It is <u>finite-gain</u> L-stable if

$$\|(Hu)_\tau\|_L \leq \gamma \|u_\tau\|_L + \beta \qquad \forall u \in L_e^m, \ \forall \tau \geq 0$$

- where γ, β are positive constants.

- The smallest possible value for γ is called <u>gain</u>.
- The norms can be L_p for any $p \in [1,\infty]$ in that case, we say H is L_p-stable

Example:

$$\dot{x} = -x + u, \quad x(0) = x_0$$
$$y = x$$

$$\Longrightarrow \quad x(t) = x_0 e^{-t} + \int_0^t e^{-(t-s)} u(s)\,ds$$

$$\Longrightarrow \quad \|x(t)\| \leq e^{-t}\|x_0\| + \int_0^t e^{-(t-s)} \|u(s)\|\,ds$$

$$\leq e^{-t}\|x_0\| + \|u\|_{L_\infty} \underbrace{\int_0^t e^{-(t-s)}\,ds}_{\leq 1}$$

$$\leq e^{-t}\|x_0\| + \|u\|_{L_\infty}$$

$$\Longrightarrow \quad \|y\|_{L_\infty} = \|x\|_{L_\infty} \leq \|u\|_{L_\infty} + \|x_0\|$$

$\Longrightarrow$ H is L_∞-stable with finite gain $\gamma = 1$

$\beta = \|x_0\|$

Now, if $y = x^2$, then

$$\|y\|_{L_\infty} = \|x\|^2_{L_\infty} \leq (\|x_0\| + \|u\|_{L_\infty})^2$$

$\frac{1}{2}(a+b)^2 \leq a^2 + b^2$

$$\leq \underbrace{2\|u\|^2_{L_\infty}}_{\alpha(\|u\|_{L_\infty})} + \underbrace{2\|x_0\|^2}_{\beta} \longrightarrow \text{No finite gain}$$

$\rightsquigarrow \alpha(r) = 2r^2$

- What should we do in general? $\longleftrightarrow$ Lyapunov functions.

Example: $$\dot{x} = -x - x^3 + u$$
$$y = \tanh(x) + u$$

- $V(x) = x^2 \implies \dot{V}(x) = 2x(-x - x^3 + u)$
$$\leq -2x^2 + 2xu$$
$$\leq -2V(x) + 2\sqrt{V(x)}\,|u|$$

- Let $W(x) = \sqrt{V(x)}$, then, $\dot{W} = \dfrac{\dot{V}}{2\sqrt{V}}$

$$\implies \dot{W}(x) \leq -W(x) + |u|$$

Comparison-lemma
$$\implies W(x(t)) \leq e^{-t} W(x(0)) + \int_0^t e^{-(t-s)} |u(s)|\,ds$$
$$\implies |x(t)| \leq e^{-t} |x_0| + \int_0^t e^{-(t-s)} |u(s)|\,ds$$
$$\leq |x_0| + \|u\|_{L_\infty}$$
$$\implies |y(t)| \leq |\tanh(x(t))| + |u(t)|$$

$|\tanh(x)| \leq |x|$ $\longleftarrow$
$$\leq |x(t)| + |u(t)|$$
$$\leq |x_0| + 2\|u\|_{L_\infty} \implies \|y\|_{L_\infty} \leq 2\|u\|_{L_\infty} + |x_0|$$

Thm (L-Stability with Lyapunov func) Thm. 5.1

for globally exp. stable case

Consider,

$$\dot{x} = f(x,u)$$

$$y = h(x,u)$$

- assume $x=0$ is eqlb. point, $f(0,0)=0$
- assume there exists a Lyapunov funct. s.t.

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2$$

$$\dot{V}(x) \leq -c_3 \|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\|$$

- assume

$$\|f(x,u) - f(x,0)\| \leq L \|u\|$$

$$\|h(x,u)\| \leq \eta_1 \|x\| + \eta_2 \|u\|$$

- Then,

$$\Longrightarrow \quad \|y\|_{L_p} \leq \gamma \|u\|_{L_p} + \beta ,$$

where $\gamma = \eta_2 + \frac{\eta_1 c_2 c_4 L}{c_1 c_3}$, $\beta = \eta_1 \|x_0\| \sqrt{\frac{c_2}{c_1}}\, \rho$

$$\rho = \begin{cases} 1 & p=\infty \\ \left(\frac{2c_2}{c_3 p}\right)^{\frac{1}{p}} & p \in [1,\infty) \end{cases}$$

Let's check the conditions of the theorem for our example:

$$\dot{x} = -x - x^3 + u =: f(x,u)$$
$$y = \tanh(x) + u =: h(x,u)$$

- $f(0,0) = 0$ ✓

- $V(x) = x^2 \implies$

$$x^2 \leq V(x) \leq x^2 \quad ✓ \qquad c_1 = c_2 = 1$$

$$\frac{\partial V}{\partial x} f(x,0) = 2x(-x-x^3) \leq -2x^2 \quad ✓ \qquad c_3 = -2$$

$$\left\| \frac{\partial V}{\partial x} \right\| = |2x| \leq 2|x| \quad ✓ \qquad c_4 = 2$$

- $|f(x,u) - f(x,0)| = |u|$ ✓ $\qquad L = 1$

$$|h(x,u)| \leq |\tanh(x)| + |u| \leq |x| + |u| \quad ✓ \qquad \eta_1 = 1, \ \eta_2 = 1$$

- The notion of L-stability depends on the $\|\cdot\|_L$ norm in definition.

- For many systems of interest, if system is L_p-stable for certain "p", it is stable for any other p
  - linear systems
  - systems that satisfy conditions of the theorem.

- But, there examples that are L_p-stable in one "p", but unstable for other "p"

example: $y(t) = \sqrt{u(t)}$

$$\|y\|_{L_\infty} \leq \sqrt{\|u\|_{L_\infty}} \Rightarrow L_\infty\text{-stable}$$

with $\alpha(r) = \sqrt{r}$ and $\beta = 0$

but not L_1-stable

$$u(t) = \left(\frac{1}{1+t}\right)^2 \Rightarrow \|u\|_{L_1} = \int_0^\infty \frac{1}{(1+t)^2} dt = \frac{-1}{1+t}\Big|_0^\infty = 1$$

$$\text{but } \|y\|_{L_1} = \int_0^\infty \frac{1}{1+t} dt = \log(t)\Big|_0^\infty = \infty$$